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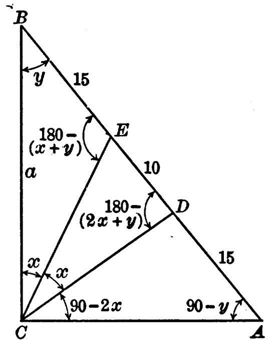
SOLUTION BY MARCIA LATHAM, New York City.

Let $BC = a$, $AC = b$, $DC = c$, $\angle DCE = \angle ECB = x$, $\angle ABC = y$. Then $\angle DCA = 90 - 2x$, $\angle EDC = 180 - (2x + y)$, and $\angle BEC = 180 - (x + y)$.



$$\sin y = b/40 \qquad \text{and} \qquad \cos y = a/40. \tag{1}$$

Since EC bisects $\angle BCD$,

$$a/c = 15/10 = 3/2. \tag{2}$$

In the triangle, DCA, by the law of sines,

$$c/15 = \cos y/\cos 2x. \tag{3}$$

Combining (1), (2), and (3), $2a/45 = a/40 \cos 2x$, or $\cos 2x = 9/16$, $x = 1/2 \cos^{-1} 9/16 = 27° 53' 8''$. In the triangle, BCD, by law of sines, $\sin(2x + y)/\sin y = a/c$. Whence, by (1) and (2)

$$\sin(2x + y) = 3b/80. \tag{4}$$

Also, $a/25 = \sin(2x + y)/\sin 2x$; whence, by (4), $\sin 2x = 15b/16a$. But $\sin^2 2x + \cos^2 2x = 1$. Then $(15b/16a)^2 + (9/16)^2 = 1$; whence

$$b^2 = 7a^2/9. \tag{5}$$

Now, in the triangle, ABC, $a^2 + b^2 = (40)^2$; whence, by (5), $a^2 + 7a^2/9 = 1{,}600$; whence, $a = 30$ and from (5), $b = 10\sqrt{7}$.

Also solved by A. M. HARDING, POLYCARP HANSEN, C. E. HORNE, R. A. JOHNSON, ELMER LATSHAW, E. W. MARTIN, LOUIS ORDANKSY, A. PELLETIER, J. L. RILEY, H. M. ROESER, L. SMITH, D. L. STAMY, H. TSAI, and L. G. WELD.

2760 [1919, 124]. Proposed by CHARLES N. SCHMALL, New York City.

In an arithmetical progression, if s_n be the sum of the first n terms, s_{2n} the sum of the first $2n$ terms, and s_{3n} the sum of the first $3n$ terms of the same series, prove that $s_{2n} - s_n = \frac{1}{3}s_{3n}$.

SOLUTION BY EMMA M. GIBSON, Springfield (Mo.) High School.

The sum of n terms of an arithmetical progression is expressed by the formula

$$s_n = \frac{n(a_1 + a_n)}{2},$$

where a_1 and a_n are the first and nth terms, respectively.

Hence, $s_n = n(a_1 + a_n)/2$, $s_{2n} = 2n(a_1 + a_{2n})/2$, and $s_{3n} = 3n(a_1 + a_{3n})/2$ are the sums of the first n terms, the first $2n$ terms, and the first $3n$ terms, respectively. Now $a_{2n} = a_n + nd$, $a_{3n} = a_{2n} + nd = a_n + 2nd$, d being the common difference.

Then $s_{2n} = 2n(a_1 + a_n + nd)/2$ and $s_{3n} = 3n(a_1 + a_n + 2nd)/2$ and

$$s_{2n} - s_n = 2n(a_1 + a_n + nd)/2 - n(a_1 + a_n)/2 = n(a_1 + a_n + 2nd)/2 = s_{3n}/3.$$

Also solved by R. D. BOHANNAN, H. L. BRIDGES, JR., H. N. CARLETON, W. F. CHENEY, JR., P. J. DA CUNHA, H. C. GOSSARD, WILLIAM HERBERG, C. N. MILLS, LOUIS O'SHAUGHNESSEY, H. L. OLSON, A. PELLETIER, J. B. REYNOLDS, I. S. SUN, and ELIJAH SWIFT.

2768 [1919, 171]. Proposed by PAUL CAPRON, U. S. Naval Academy.

Given the center, a focus, and a point of a conic, construct geometrically the circle of curvature at the point.

I. SOLUTION BY P. J. DA CUNHA, University of Lisbon.

Soient F le foyer donné, C le centre, F' l'autre foyer, determiné sur FC par $CF' = FC = c$, $2a$ et $2b$ les axes de la conique, $e = \sqrt{a^2 \pm b^2}/a$ son excentricité, $2p$ son paramètre, M le point donné, δ et δ' les rayons vecteurs FM et $F'M$, R et n le rayon de courbure et la longueur de la normale MT à la conique au même point, 2ω l'angle FMF' pour l'ellipse et son supplément pour l' hyperbole, x, y les coordonnées de M, en supposant la conique rapportée à ses axes.

En considérant d'abord le cas d'ellipse, nous partirons des formules connues

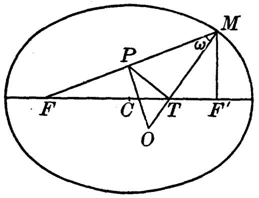
$$R = \frac{n^3}{p^2}, \qquad n = \frac{1}{a^2}\sqrt{a^4y^2 + b^4x^2}, \qquad \delta = a + ex, \qquad \delta' = a - ex, \tag{1}$$

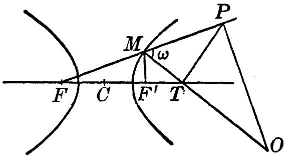
d'où il résulte, en premier lieu

$$n^2 = \frac{a^4y^2 + b^4x^2}{a^4} = \frac{a^2(a^2b^2 - b^2x^2) + b^4x^2}{a^4} = \frac{a^4b^2 - b^2(a^2 - b^2)x^2}{a^4} = \frac{b^2(a^2 - e^2x^2)}{a^2} = \frac{b^2\delta\delta'}{a^2}$$

ou

$$\delta\delta' = \frac{a^2}{b^2}n^2. \tag{2}$$





Le triangle FMF' donne, d'autre part, $4c^2 = \delta^2 + \delta'^2 - 2\delta\delta' \cos 2\omega = (\delta + \delta')^2 - 4\delta\delta' \cos^2 \omega$ ou, en égard à l'équation (2) $4c^2 = 4a^2 - (4a^2/b^2)n^2 \cos^2 \omega$, d' où il vient, toutes réductions faites, $n \cos \omega = b^2/a = p$. La première des formules (1) donne ensuite

$$R = \frac{n}{\cos^2 \omega}. \tag{3}$$

Un calcul analogue, appliqué à l'hyperbole, nous conduirait au même résultat. Or la formule (3) résout le problème, parce qu' elle donne immédiatement, la construction suivante:

Par le point T, où la normale coupe l'axe FF', on mène une perpendiculaire à MT, et par le point P, ou elle coupe FM, on mène une autre perpendiculaire à FM. Le point de rencontre O de cette droite avec MT est le centre de courbure demandé.

Obs. L'énoncé du problème suppose une conique à centre, mais il est facile de vérifier que le résultat obtenu s'applique également à la parabole.

Note. La solution ci-dessus n'est pas nouvelle. Voir Duhamel, *Eléments de calcul infinitesimal*, tome 1, Paris, 1856, p. 172–173.

II. SOLUTION BY GRACE M. BAREIS, OHIO STATE UNIVERSITY.

Let the center, focus and point of the conic be designated by C, F and M, respectively. On the line through CF, locate F', the other focus, so that $F'C = CF$. Describe the circle through F', F, M. Join M to the ends of the diameter that passes through C. These two lines are the tangent and normal to the conic at M. The conic is a hyperbola or an ellipse according as the line that divides the segment CF internally or the one that does not so divide it is taken as tangent. The problem, therefore, has two solutions.

Let L and T be the points in which the tangent and normal, respectively, cut the line joining C and F. Let L' be the foot of the perpendicular from M on CF; then, L' and L are conjugate points with respect to the conic. Now find A and A', the double points in the involution determined by the pair of points L and L' and the center of the involution C. These points A and A' are the vertices of the conic. The lines through A and A', respectively, and perpendicular to CF

are the tangents at these points. The conic is, therefore, determined; for we have three points M, A, A' and the tangent at each of them, known.

From this point on the construction may be completed in any one of three ways:

(1) By the use of Pascal's Theorem, find the point H in which the normal at M cuts the conic again. Draw the circle on HM as diameter. Find the fourth point of intersection G of this circle with the conic. Draw HG. Through M draw MK parallel to HG and determine the point K in which this line cuts the conic again. The circle tangent to the conic at M and passing through K is the required circle of curvature at M. (Cf. L. Cremona, *Elements of Projective Geometry*, Oxford, 1885, p. 190).

(2) By the method previously described both the ellipse and the hyperbola that have center C, focus F, and point M are determined. The construction may, therefore, be completed by making use of the fact that at the intersection of two confocal conics the center of curvature of either is the pole with respect to the other of the tangent to the former at the intersection.

(3) Through F, draw a line parallel to the tangent at M. By Desargues' Theorem, find the points R and R' in which this line meets the conic. On MF take Q so that $MQ = RR'$. Then the circle through Q and tangent to the conic is the required circle of curvature; for, the focal chord of curvature is equal to the focal chord of the conic drawn parallel to the tangent at the point. For results in (2) and (3) the reader may consult Salmon's *Conic Sections*, 6th ed., 1879, pp. 374–376.

It may be noted that if C is at infinity, the conic becomes a parabola with the axis CF, the focus F, and the point M given. Two parabolas may be drawn satisfying these conditions according to which of the bisectors of the angle made by MF and the diameter through M is taken as tangent. The above methods of construction then apply to this case also.

Also solved by H. HALPERIN, A. PELLETIER, J. B. REYNOLDS, and the Proposer.

2771 [1919, 191]. Proposed by GEORGE PAASWELL, New York City.

A circle is revolved through an angle of 90° about a vertical chord which does not pass through the center of the circle. Taking the origin at the lower extremity of the chord, the z-axis along the chord and the x- and y-axes in the boundary planes, pass a plane through the x-axis making a given angle with the xy plane. Determine the portion of the area of the surface above the plane and between the xz and yz planes.

I. SOLUTION BY J. B. REYNOLDS, Lehigh University.

A solution by vector analysis. Let $r = a\cos\theta + b\sin\theta$ be the polar equation of the circle revolved. From this we see that for a vector equation of the surface generated, we may write

$$r = (a\cos t + b\sin t)\cos t\cos u\cdot i + (a\cos t + b\sin t)\cos t\sin u\cdot j + (a\cos t + b\sin t)\sin t\cdot k;$$

or

$$r = \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\cos u\cdot i + \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\sin u\cdot j + \tfrac{1}{2}\{b + a\sin 2t - b\cos 2t\}\cdot k$$

from which

$$\frac{\partial r}{\partial t} = \{-a\sin 2t + b\cos 2t\}\cos u\cdot i + \{-a\sin 2t + b\cos 2t\}\sin u\cdot j + \{a\cos 2t + b\sin 2t\}\cdot k,$$

$$\frac{\partial r}{\partial u} = -\tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\sin u\cdot i + \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\cos u\cdot j,$$

giving the vector product

$$\begin{aligned}\frac{\partial r}{\partial t}\times\frac{\partial r}{\partial u} = &-\tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\{a\cos 2t + b\sin 2t\}\cos u\cdot i\\ &-\tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\{a\cos 2t + b\sin 2t\}\sin u\cdot j\\ &+\tfrac{1}{2}\{-a\sin 2t + b\cos 2t\}\{a + a\cos 2t + b\sin^2 2t\}\cdot k.\end{aligned}$$

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[1] Of course it is unnecessary to describe the circle through F', F, M, for the tangent and normal bisect angles given at M; nor is it necessary to invoke an involution in order to get A and A' since we have given $F'M + FM = 2a$.—EDITOR.